# The Konyagin–Shkredov Sum-Product Bound

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## 1 Introduction

For a finite set $A \subset \mathbb{R}$, the *sum set* of $A$ is defined as

$$A + A = \{a + a' : a, a' \in A\}.$$

Similarly, we define the *difference set*, *product set*, and *ratio set* of $A$ respectively as

$$A - A = \{a - a' : a, a' \in A\}, \quad AA = \{a \cdot a' : a, a' \in A\}, \text{ and } A/A = \{a/a' : a, a' \in A\}.$$

Solymosi [8] derived the following result.[1]

**Theorem 1.1.** *Let $A \subset \mathbb{R}$ be a finite set. Then* $\max\{|A + A|, |AA|\} = \Omega^*(|A|^{4/3})$.

To prove Theorem 1.1, Solymosi proved

$$|A + A|^2 |AA| = \Omega^* \left(|A|^4\right). \tag{1}$$

Note that Theorem 1.1 is an immediate corollary of (1). While Theorem 1.1 is conjectured to be far from tight, the bound (1) is tight. By taking $A$ to be an arithmetic progression we have $|A + A| = \Theta(|A|)$ and $|AA| = \Theta(|A|^2 \lg^\beta |A|)$ with $\beta \approx -0.086$ (this result is attributed to Erdős in [1]).

Konyagin and Shkredov [4, 5] improved the bound of Theorem 1.1 by studying what happens when (1) is close to being tight. In this document we describe in detail the proof of Konyagin and Shkredov. We assume that the reader is familiar with the proof of Theorem 1.1, and especially with the concept of energy (for detailed notes by the author, see [6]).

[1] In the $\Omega^*(\cdot)$, $O^*(\cdot)$, and $\Theta^*(\cdot)$ notation we ignore subpolynomial factors; that is, factors that are asymptotically smaller than $|A|^\varepsilon$ for every $\varepsilon > 0$.

**Theorem 1.2 (Konyagin and Shkredov [5]).** *Consider a finite set $A \subset \mathbb{R}$. Then for any $c < 5/9813$ we have*

$$\max\{|A+A|, |AA|\} = \Omega^*(|A|^{4/3+c}).$$

We begin with a few standard definitions. The *multiplicative energy* of a finite set $A \subset \mathbb{R}$ is

$$E^{\times}(A) = \left|\left\{(a_1, a_2, a_3, a_4) \in A^4 : a_1 a_2 = a_3 a_4\right\}\right|.$$

For any $x \in AA$ we set $r_A^{\times}(x) = |\{(a_1, a_2) \in A^2 : a_1 a_2 = x\}|$. Since every pair in $A \times A$ participates in exactly one $r_A^{\times}(x)$, we have $\sum_{x \in AA} r_A^{\times}(x) = |A|^2$. By the Cauchy-Schwarz inequality, we have

$$E^{\times}(A) = \sum_{x \in AA} r_A^{\times}(x)^2 \geq \frac{\left(\sum_x r_A^{\times}(x)\right)^2}{|AA|} = \frac{|A|^4}{|AA|}. \tag{2}$$

We similarly define the *additive energy* of $A$ as

$$E^{+}(A) = \left|\left\{(a_1, a_2, a_3, a_4) \in A^4 : a_1 + a_2 = a_3 + a_4\right\}\right|.$$

For any $x \in A+A$ we set $r_A^{+}(x) = |\{(a_1, a_2) \in A^2 : a_1 + a_2 = x\}|$. Since every pair $(a, a') \in A^2$ contributes 1 to $r_A^{+}(a + a')$ we have $\sum_{x \in A+A} r_A^{+}(x) = |A|^2$. By the Cauchy-Schwarz inequality

$$E^{+}(A) = \sum_{x \in A+A} r_A^{+}(x)^2 \geq \frac{\left(\sum_{x \in A+A} r_A^{+}(x)\right)^2}{|A+A|} = \frac{|A|^4}{|A+A|}. \tag{3}$$

# 2 Connecting $E^+(A)$ and $|AA|$

In this section we begin to compare the additive properties and multiplicative properties of a set $A$, by deriving a connection between $E^+(A)$ and $|AA|$. The connection is obtained by double counting the number of collinear triples in a lattice. This technique was introduced by Elekes and Ruzsa [1] to show that a small sum-set implies a large ratio set.

First, we present a couple of incidence results. Given a point set $\mathcal{P}$ and a set of lines $\mathcal{L}$, both in $\mathbb{R}^2$, we say that an *incidence* is a pair $(p, \ell) \in \mathcal{P} \times \mathcal{L}$ with the point $p$ being on the line $\ell$. We denote by $I(\mathcal{P}, \mathcal{L})$ the number of incidences in $\mathcal{P} \times \mathcal{L}$. Figure 1 depicts a configuration with nine point-line incidences.

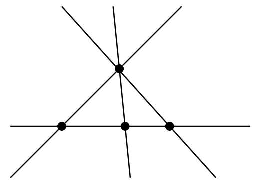

Figure 1: A configuration of four points, four lines, and nine incidences.

**Theorem 2.1 (The Szemerédi-Trotter theorem [9]).** *Let $\mathcal{P}$ be a set of $m$ points and let $\mathcal{L}$ be a set of $n$ lines, both in $\mathbb{R}^2$. Then $I(\mathcal{P},\mathcal{L}) = O(m^{2/3}n^{2/3} + m + n)$.*

We rely on Theorem 2.1 to prove the following.

**Corollary 2.2.** *(a) Let $\mathcal{P}$ be a set of $m$ points in $\mathbb{R}^2$ and let $k$ be a positive integer. The number of lines in $\mathbb{R}^2$ that are incident to at least $k$ points of $\mathcal{P}$ is $O\left(\frac{m^2}{k^3} + \frac{m}{k}\right)$.*
*(b) Let $A$ be a set of $m$ real numbers. Then the number of collinear triples in $A \times A \subset \mathbb{R}^2$ is $O(m^4 \lg m)$.*

*Proof.* We first prove part (a). As a trivial bound we have $O(m^2)$ lines for every $k \geq 2$, since at most $\binom{m}{2}$ lines pass through at least two points of $\mathcal{P}$. When $k$ is a constant part (a) is obtained by taking a sufficiently large constant in the $O(\cdot)$-notation. We may thus assume that $k$ is larger than the constant in the $O(\cdot)$-notation of Theorem 2.1.

Let $\mathcal{L}$ denote the set of lines that are incident to at least $k$ points of $\mathcal{P}$, and set $n_k = |\mathcal{L}|$. By definition, we have $I(\mathcal{P},\mathcal{L}) \geq n_k k$. On the other hand, Theorem 2.1 implies $I(\mathcal{P},\mathcal{L}) = O(m^{2/3}n_k^{2/3} + n_k + m)$. Combining these two bounds yields $n_k k = O(m^{2/3}n_k^{2/3} + n_k + m)$. Since we assume that $k$ is larger than the constant in the $O(\cdot)$-notation, the dominating term on the right-hand side cannot be $n_k$. That is, we get that $n_k k = O(m^{2/3}n_k^{2/3} + m)$, which immediately implies $n_k = O\left(\frac{m^2}{k^3} + \frac{m}{k}\right)$.

We move to prove part (b) of the corollary. As before, for any $k \geq 3$ we denote by $n_k$ the number of lines that contain at least $k$ points of $A \times A$. By combining part (a) of the corollary with a dyadic decomposition, we get that the number of collinear triples is at most

$$\sum_{j=3}^{m} \binom{j}{3} n_j \leq \sum_{k=1}^{\lg m} \binom{2^{k+1}}{3} n_{2^k+1} = \sum_{k=1}^{\lg m} \binom{2^{k+1}}{3} \cdot O\left(\frac{m^4}{(2^k+1)^3} + \frac{m^2}{2^k+1}\right)$$
$$= \sum_{k=1}^{\lg m} O\left(m^4 + m^2 2^{2k}\right) = O\left(m^4 \lg m\right). \qquad \square$$

We are now ready to prove the main result of this section.

**Theorem 2.3.** *Let $A$ be a finite set of positive real numbers. Then*

$$E^+(A)^4 = O(|AA||A|^{10} \lg |A|).$$

*Proof.* We consider the set of elements of $A+A$ that have many representations:

$$F = \left\{ x \in A+A : \, r_A^+(x) > \frac{E^+(A)}{2|A|^2} \right\}.$$

Notice that $F$ also consists of positive real numbers. We have that

$$\sum_{x \in F} r_A^+(x)^2 = \sum_{x \in A+A} r_A^+(x)^2 - \sum_{\substack{x \in A+A \\ x \notin F}} r_A^+(x)^2 \geq E^+(A) - |A|^2 \cdot \frac{E^+(A)}{2|A|^2} = \frac{E^+(A)}{2}.$$

That is, the elements of $F$ yield most of the additive energy of $A$. We set $r_A^+(F) = \sum_{x \in F} r_A^+(x)$. Since every $x \in A+A$ satisfies $r_A^+(x) \leq |A|$, we get that

$$r_A^+(F) = \sum_{x \in F} r_A^+(x) \geq \sum_{x \in F} \frac{r_A^+(x)^2}{|A|} \geq \frac{E^+(A)}{2|A|}.$$

Moreover, by the definition of $F$ we have

$$|F| < r_A^+(F) \cdot \frac{2|A|^2}{E^+(A)}.$$

Combining the above implies

$$|F| + |A| < r_A^+(F) \cdot \frac{2|A|^2}{E^+(A)} + |A| \cdot \frac{r_A^+(F) \cdot 2|A|}{E^+(A)} = \frac{4|A|^2 r_A^+(F)}{E^+(A)}. \tag{4}$$

Consider the lattice $\mathcal{P} = (A \cup F) \times (A \cup F) \subset \mathbb{R}^2$ and let $T$ denote the number of collinear triples in $\mathcal{P}$. The proof is based on double counting $T$. By Corollary 2.2 and (4), we have

$$T = O\left((|A| + |F|)^4 \lg |A+F|\right) = O\left(\frac{|A|^8 r_A^+(F)^4}{E^+(A)^4} \lg |A|\right). \tag{5}$$

For any $a \in A$, we set $F(a) = \{a' \in A : \, a + a' \in F\}$. For a pair $(a, a') \in A^2$, we are interested in the number of quadruples $(b_1, b_2, b_1', b_2') \in F(a)^2 \times F(a')^2$ with

$b_1b_1' = b_2b_2'$. The reason is that every such quadruple corresponds to a collinear triple of the form

$$(a, a'),\ (a + b_1, a' + b_2'),\ (a + b_2, a' + b_1'). \tag{6}$$

Indeed, notice that the slope of the line through the first two points is $b_2'/b_1$ and the slope of the line through the first and third points is $b_1'/b_2$. These slopes are identical since $b_1b_1' = b_2b_2'$.

Each triple of the form that is described in (6) consists of three points of $(A \cup F) \times (A \cup F)$ and corresponds to a unique six-tuple $(a, a', b_1, b_2, b_1', b_2')$. By repeating the Cauchy-Schwarz argument that was used in (2), we obtain that the number of quadruples $(b_1, b_2, b_1', b_2') \in F(a)^2 \times F(a')^2$ with $b_1b_1' = b_2b_2'$ is at least $|F(a)|^2|F(a')|^2/|AA|$. This in turn implies

$$T \geq \sum_{a,a' \in A} \frac{F(a)^2F(a')^2}{|AA|} = \frac{\left(\sum_{a\in A} F(a)^2\right)^2}{|AA|}.$$

By applying the Cauchy-Schwarz inequality once again, we have

$$\sum_{a\in A} F(a)^2 \geq \frac{\left(\sum_{a\in A} F(a)\right)^2}{|A|} = \frac{4r_A^+(F)^2}{|A|}.$$

Combining the above yields

$$T \geq \frac{16r_A^+(F)^4}{|A|^2 \cdot |AA|}. \tag{7}$$

Combining (5) and (7) implies the assertion of the theorem. $\square$

# 3 Linear relations between three sets

For $A_1, A_2, A_3 \subset \mathbb{R}$, we set

$$\sigma(A_1, A_2, A_3) = \left|\{(a_1, a_2, a_3) \in A_1 \times A_2 \times A_3 :\ a_1 + a_2 + a_3 = 0\}\right|.$$

Given a finite set $A$ and $\lambda \in A/A$, we set $A_\lambda = A \cap (\lambda A)$. Notice that $|A_\lambda| = r_A^/(\lambda)$. For $t \in \mathbb{R}$ we define

$$\hat{A}_t = \{\lambda \in A/A :\ t \leq |A_\lambda| < 2t\}.$$

The proof of the following lemma is a variant of the proof of Theorem 1.1. In this proof, $\alpha A = \{\alpha \cdot a :\ a \in A\}$.

**Lemma 3.1.** *Consider a finite set $A$ of positive real numbers and let $t$ and $\sigma$ be positive real numbers such that $32\sigma \le t^2 \le |A+A|\sqrt{\sigma}$. Moreover, let $\hat{A}'_t \subset \hat{A}_t$ such that for any nonzero $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $\lambda_1, \lambda_2, \lambda_3 \in \hat{A}'_t$ we have*

$$\sigma(\alpha_1 A_{\lambda_1}, \alpha_2 A_{\lambda_2}, \alpha_3 A_{\lambda_3}) \le \sigma. \tag{8}$$

*Then*

$$|A+A|^2 \ge |\hat{A}'_t| \frac{t^3}{16\sqrt{8\sigma}}.$$

*Proof.* Let $M$ be an integer which we will set below. We sort the elements of $\hat{A}'_t$ in decreasing order, and then split $\hat{A}'_t$ into sets of $M$ consecutive elements (discarding at most $M-1$ of the smallest elements). We denote these sets as $S_1, \ldots, S_k$ where $k = \left\lfloor \frac{|\hat{A}'_t|}{M} \right\rfloor \ge \frac{|\hat{A}'_t|}{2M}$.

We consider the lattice $\mathcal{P} = A \times A$ and double count $|\mathcal{P}+\mathcal{P}|$, where we have the trivial bound $|\mathcal{P}+\mathcal{P}| = |(A+A)\times(A+A)| = |A+A|^2$. To derive a lower bound for $|\mathcal{P}+\mathcal{P}|$, we rely on several observations from the proof of Theorem 1.1. Specifically, we again consider the lines that are incident to the origin and have a slope from $A/A$. This time we separately consider each set of slopes $S_i$, by summing up pairs of points of $\mathcal{L}$ that are on lines with slopes in $S_i$. Notice that if $p$ and $p'$ are points on lines with slopes from $S_i$ and $q$ and $q'$ are on lines with slopes from $S_j$ (where $i \ne j$), then $p+p' \ne q+q'$ ($p+p'$ is in the wedge bounded by the lines with slopes of $S_i$ and $q+q'$ is in the wedge bounded by the lines with slopes from $S_j$).

Setting $A_{\lambda,2} = \{(a,a') \in A : a'/a = \lambda\}$, we have

$$|\mathcal{P}+\mathcal{P}| \ge \sum_{j=1}^{k} \left( t^2 \binom{M}{2} - \sum_{\substack{\lambda_1,\ldots,\lambda_4 \in S_j,\, \lambda_1 \ne \lambda_2, \\ \lambda_3 \ne \lambda_4,\, (\lambda_1,\lambda_2) \ne (\lambda_3,\lambda_4)}} |(A_{\lambda_1,2} + A_{\lambda_2,2}) \cap (A_{\lambda_3,2} + A_{\lambda_4,2})| \right). \tag{9}$$

We will now bound the value of the sum in (9) that goes over $\lambda_1, \ldots, \lambda_4 \in S_j$. Consider one element of the sum, with fixed values for $\lambda_1, \ldots, \lambda_4$. By the restrictions on $\lambda_1, \ldots, \lambda_4$, at least one of these four ratios differs from the other three. Without loss of generality, we assume that $\lambda_4 \notin \{\lambda_1, \lambda_2, \lambda_3\}$.

Consider

$$z = (z_1, z_2) \in (A_{\lambda_1,2} + A_{\lambda_2,2}) \cap (A_{\lambda_3,2} + A_{\lambda_4,2}).$$

Then for some $a_1 \in A_1, \ldots, a_4 \in A_4$ we have $z_1 = a_1 + a_2 = a_3 + a_4$ and $z_2 = \lambda_1 a_1 + \lambda_2 a_2 = \lambda_3 a_3 + \lambda_4 a_4$. This implies

$$0 = \lambda_1 a_1 + \lambda_2 a_2 - \lambda_3 a_3 - \lambda_4 a_4 - \lambda_4(a_1 + a_2 - a_3 - a_4)$$

(the expression in the parentheses to the right equals zero). After rearranging this equation, we get

$$(\lambda_1 - \lambda_4)a_1 + (\lambda_2 - \lambda_4)a_2 - (\lambda_3 - \lambda_4)a_3 = 0. \tag{10}$$

Since $\lambda_4 \notin \{\lambda_1, \lambda_2, \lambda_3\}$, none of the expressions in the parentheses equals zero, so by (8) we have at most $\sigma$ solutions to (10). That is, $|(A_{\lambda_1,2}+A_{\lambda_2,2})\cap(A_{\lambda_3,2}+A_{\lambda_4,2})| \leq \sigma$. Combining this with (9) and recalling that $k \geq \frac{|\hat{A}'_t|}{2M}$ yields

$$|\mathcal{P}+\mathcal{P}| \geq k\left(t^2\binom{M}{2} - \sigma M^4\right) > \frac{|\hat{A}'_t|}{2M}\left(\frac{t^2M^2}{4} - \sigma M^4\right).$$

To optimize this bound, we set $M = \lceil t/\sqrt{8\sigma}\rceil$. From the assumption $32\sigma \leq t^2$, we get that $M \geq 2$. By combining this with the straightforward $|\mathcal{P}+\mathcal{P}| = |A+A|^2$, we obtain

$$|A+A|^2 \geq |\hat{A}'_t|\left(\frac{t^2M}{8} - \frac{\sigma M^3}{2}\right) \geq |\hat{A}'_t|\frac{t^3}{16\sqrt{8\sigma}}.$$

Notice that if $M > |\hat{A}'_t|$ then we cannot partition $|\hat{A}'_t|$ into $M$ parts, and the above analysis fails. To handle this case we assume for contradiction that $M > |\hat{A}'_t|$ and the assertion of the lemma fails. That is, we have

$$|A+A|^2 < |\hat{A}'_t|\frac{t^3}{16\sqrt{8\sigma}} < M\frac{t^3}{16\sqrt{8\sigma}} \leq \frac{t^4}{128\sigma}.$$

This contradicts the assumption $t^2 \leq |A+A|\sqrt{\sigma}$ and completes the proof. $\square$

We now define the additive energy of two finite sets $A, B \subset \mathbb{R}$. Similarly to $r^+_A(x)$ and $E^+(A)$, we define

$$r^+_{A,B}(x) = |\{(a,b) \in A \times B :\, a+b = x\}|, \qquad \text{and}$$
$$E^+(A,B) = |\{(a,a',b,b') \in A^2 \times B^2 :\, a+b = a'+b'\}|.$$

Notice that $E^+(A,B) = \sum_{x\in A+B} r^+_{A,B}(x)^2$. Moreover, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} E^+(A,B) &= |\{(a,a',b,b') \in A^2 \times B^2 :\, a-a' = b'-b\}| = \sum_{x\in A} r^-_A(x)r^-_B(x) \\ &\leq \sqrt{\sum_{x\in A-A} r^-_A(x)^2 \cdot \sum_{y\in A-A} r^-_B(y)^2} \leq E^+(A)^{1/2}E^+(B)^{1/2}. \end{aligned} \tag{11}$$

**Lemma 3.2.** *Consider a finite set $A$ of positive real numbers, and let $r \geq 1$ be a real number such that $|A+A|^2|AA| \leq r|A|^4$. Then there exist $t > E^\times(A)/(2|A|^2)$ and $\hat{A}'_t \subset \hat{A}_t \subset A/A$ such that $|\hat{A}_t| = \Omega(E^\times(A)/t^2 \lg |A|)$, $|\hat{A}'_t| \geq |\hat{A}_t|/2$, and for any $\lambda \in \hat{A}'_t$ we have*

$$E^+(A_\lambda) = \Omega(t^3/r^4 \lg^4 |A|),$$

*and*

$$|A_\lambda A_\lambda| = \Omega(t^2/r^{16} \lg^{17} |A_\lambda|).$$

*Proof.* We have

$$\sum_{\substack{x \in AA \\ r_A^\times(x) \leq E^\times(A)/2|A|^2}} r_A^\times(x)^2 \leq \frac{E^\times(A)}{2|A|^2} \sum_{\substack{x \in AA \\ r_A^\times(x) \leq E^\times(A)/2|A|^2}} r_A^\times(x) \leq E^\times(A)/2.$$

Thus, by a dyadic pigeonholing:

$$E^\times(A)/2 \leq \sum_{\substack{x \in AA \\ r_A^\times(x) > E^\times(A)/2|A|^2}} r_A^\times(x)^2 = \sum_{j=\lg \frac{E^\times(A)}{2|A|^2}}^{\lg |A|} \sum_{\substack{x \in AA \\ 2^j \leq r_A^\times(x) < 2^{j+1}}} r_A^\times(x)^2.$$

This implies that there exists $\lg \frac{E^\times(A)}{2|A|^2} < j \leq \lg |A|$ with

$$\left|\left\{x \in A/A :\, 2^j \leq r_A^\times(x) < 2^{j+1}\right\}\right| > E^\times(A)/2^{2(j+1)} \lg |A|.$$

That is, for $t = 2^j$ we have $|\hat{A}_t| = \Omega\left(E^\times(A)/t^2 \lg |A|\right)$, as asserted in the statement of the lemma.

We sort the elements of $\hat{A}_t$ in decreasing order according to the size of $E^+(A_\lambda)$, let $\hat{A}'_t \subset \hat{A}_t$ be the set of the first $\lceil |\hat{A}_t|/2 \rceil$ elements of $\hat{A}_t$, and set $\hat{A}''_t = \hat{A}_t \setminus \hat{A}'_t$. In the special case of $|\hat{A}_t| = 1$ we instead set $\hat{A}_t = \hat{A}'_t = \hat{A}''_t$. We will prove that there exists $\lambda \in \hat{A}''_t$ with $E^+(A_\lambda) = \Omega(t^3/r^4)$, which would in turn imply the same bound for every $\lambda \in \hat{A}'_t$.

We set $\sigma = \max_{\lambda \in \hat{A}''_t} \sqrt{2tE^+(A_\lambda)}$. Let $\lambda_1, \lambda_2, \lambda_3 \in \hat{A}''_t$ and let $\alpha, \beta \neq 0$. By the

Cauchy-Schwarz inequality and (11), we have

$$\begin{aligned}\sigma\left(A_{\lambda_1}, \alpha A_{\lambda_2}, \beta A_{\lambda_3}\right) &= \sum_{a_2 \in A_{\lambda_2}} \left|\{(a_1, a_3) \in A_{\lambda_1} \times A_{\lambda_3} : \, a_1 + \alpha a_2 + \beta a_3 = 0\}\right| \\ &\le |A_{\lambda_2}|^{1/2} \left( \sum_{a_2 \in A_{\lambda_2}} r_{A_{\lambda_1}, \beta A_{\lambda_3}}(-\alpha a_2)^2 \right)^{1/2} \\ &\le |A_{\lambda_2}|^{1/2} E^+(A_{\lambda_1}, \beta A_{\lambda_3})^{1/2} \\ &< (2t)^{1/2} E^+(A_{\lambda_1})^{1/4} E^+(A_{\lambda_3})^{1/4} \le \sigma.\end{aligned}$$

If $32\sigma \le t^2 \le |A+A|\sqrt{\sigma}$ then we may apply Lemma 3.1 with $\hat{A}''_t$, to obtain

$$|A+A|^2 \ge |\hat{A}''_t| \frac{t^3}{16\sqrt{8\sigma}} > |\hat{A}_t| \frac{t^3}{64\sqrt{8\sigma}}.$$

By rearranging and recalling that $|A+A|^2|AA| \le r|A|^4$, we get

$$\sigma > |\hat{A}_t|^2 \frac{t^6}{2^{15}|A+A|^4} \ge |\hat{A}_t|^2 \frac{t^6 |AA|^2}{2^{15} r^2 |A|^8}.$$

Since $|\hat{A}_t| = \Omega\left(E^{\times}(A)/t^2 \lg |A|\right)$ and by applying (2), we have

$$\sigma = \Omega\left( \left( \frac{|A|^4}{|AA| \cdot t^2 \cdot \lg |A|} \right)^2 \cdot \frac{t^6 |AA|^2}{r^2 |A|^8} \right) = \Omega\left( \frac{t^2}{r^2 \lg^2 |A|} \right). \tag{12}$$

On the other hand, if $32\sigma > t^2$ then we again obtain (12) (for now we ignore the case of $t^2 > |A+A|\sqrt{\sigma}$). By the definition of $\sigma$, there exists $\lambda \in \hat{A}''_t$ such that

$$E^+(A_\lambda) = \sigma^2/2t = \Omega\left( \frac{t^3}{r^4 \lg^4 |A|} \right). \tag{13}$$

By the definitions of $\hat{A}'_t$ and $\hat{A}''_t$, we get that (13) applies also to every $\lambda \in \hat{A}'_t$, which completes most of the assertions of the lemma.

Finally, for every $\lambda \in \hat{A}'_t$, combining (13) with Theorem 2.3 yields

$$\frac{t^{12}}{r^{16} \lg^{16} |A_\lambda|} = O\left( |A_\lambda A_\lambda| |A_\lambda|^{10} \lg |A_\lambda| \right),$$

or

$$|A_\lambda A_\lambda| = \Omega\left(\frac{t^{12}}{|A_\lambda|^{10} r^{16} \lg^{17}|A_\lambda|}\right) = \Omega\left(\frac{t^2}{r^{16}\lg^{17}|A_\lambda|}\right).$$

To conclude the proof, it remains to consider the case of $t^2 > |A+A|\sqrt{\sigma}$. By the definition of $\sigma$ and (3), we have

$$\sigma^2 = \max_{\lambda \in \hat{A}_t''} 2tE^+(A_\lambda) \geq \max_{\lambda \in \hat{A}_t''} 2t\frac{|A_\lambda|^4}{|A_\lambda + A_\lambda|} \geq \frac{2t^5}{|A+A|}.$$

This implies $t^8 > |A+A|^4\sigma^2 \geq 2|A+A|^3 t^5$, which contradicts $t \leq |A|$. Thus, this case cannot occur. $\square$

# 4 Szemerédi-Trotter type

We say that a finite set $A \subset \mathbb{R}$ is *SzT-type* with parameter $C(A)$ (depending only on $A$) if for every $B \subset \mathbb{R}$ and integer $\tau \geq 1$ we have[2]

$$|\{d \in A - B : r^-_{A,B}(d) \geq \tau\}| \leq C(A)|B|^2/\tau^3.$$

For $Q, R \subset \mathbb{R}$ and integer $t \geq 1$ we set $P_t(Q,R) = \{x \in Q/R : r^/_{Q,R}(x) \geq t\}$. We are interested in the parameter

$$c(A) = \min_{\substack{t\geq 1;\, Q,R\subset \mathbb{R}\setminus\{0\} \\ A \subset P_t(Q,R);\, |A| \leq \max\{|Q|,|R|\}}} |Q|^2|R|^2/t^3. \tag{14}$$

**Lemma 4.1.** *Let $A \subset \mathbb{R}$ be a finite set. Then $A$ is SzT-type with parameter $\beta \cdot c(A)$, where $\beta$ is a sufficiently large universal constant.*

*Proof.* Let $t, Q, R$ be the variables that minimize $c(A)$ in (14). In particular, we have $A \subset P_t(Q,R)$ and $|A| \leq \max\{|Q|,|R|\}$. Without loss of generality, we assume that $|Q| \geq |R|$.

Consider an arbitrary set $B \subset \mathbb{R}$. For any integer $\tau \geq 1$ we set

$$D_\tau = \{d \in A - B : r^-_{A,B}(d) \geq \tau\}.$$

[2] In [7], there is an extra $|A|$ in this definition. Below we also revise Theorem 4.2 according to this slightly different notation. Other small changes were made since we prove the sum-product case, while [5] presents a proof for the sum-quotient case (which is almost identical).

To prove the lemma, it suffices to prove that for every $\tau \geq 1$ we have

$$|D_\tau| \leq \beta |Q|^2|R|^2|B|^2\tau^{-3}t^{-3} \tag{15}$$

We set $\sigma = \sum_{d\in D_\tau} r^-_{A,B}(d)$. By definition,

$$\sigma \geq \tau |D_\tau| \tag{16}$$

We can think of $\sigma$ as the number of solutions to the equation $d = a - b$, where $d \in D_\tau$, $a \in A$, $b \in B$. Since $A \subset P_t(Q,R)$, we get that

$$\sigma \leq t^{-1}\left|\{q/r - b = d :\ q \in Q,\, r \in R,\, b \in B,\, d \in D_\tau\}\right|. \tag{17}$$

This immediately implies $\sigma \leq t^{-1}|Q||R||B|$ (since setting values for $q$, $r$, and $b$ uniquely determines $d$). Combining this with (16) yields

$$|D_\tau| \leq t^{-1}\tau^{-1}|Q||R||B|.$$

When $t^2\tau^2 \leq \beta|Q||R||B|$ we get (15), which completes the proof. We may thus assume that $t^2\tau^2 > \beta|Q||R||B|$. We may also assume that $|B|$ is sufficiently large, since small values of $|B|$ can be handled by taking $\beta$ to be sufficiently large.

We now reformulate the problem of deriving an upper bound for $\sigma$ as an incidence problem. Let $\ell_{r,d}$ denote the line in $\mathbb{R}^2$ that is defined by $y/r - x = d$. Consider the set of points $\mathcal{P} = B \times Q \subset \mathbb{R}^2$ and the set of lines

$$\mathcal{L} = \{\ell_{r,d} :\ r \in R,\, d \in D_\tau\}.$$

By (17) we have $\sigma \leq t^{-1} \cdot I(\mathcal{P}, \mathcal{L})$. Note that $|\mathcal{P}| = |B||Q|$ and $|\mathcal{L}| = |R||D_\tau|$. Combining this with Theorem 2.1 gives

$$\begin{aligned}\sigma &= t^{-1} \cdot O(|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|)\\ &= t^{-1} \cdot O((|B||Q||R||D_\tau|)^{2/3} + |B||Q| + |R||D_\tau|).\end{aligned} \tag{18}$$

If the bound of (18) is dominated by the term $t^{-1}(|B||Q||R||D_\tau|)^{2/3}$, then combining (16) and (18) and taking $\beta$ to be sufficiently large implies $|D_\tau| \leq \beta|Q|^2|R|^2|B|^2\tau^{-3}t^{-3}$. Since this implies (15), it remains to consider the case where another term of (18) dominates the bound.

Assume that the bound of (18) is dominated by the term $|B||Q|$. That is, $(|B||Q||R||D_\tau|)^{2/3} = O(|B||Q|)$, or equivalently

$$|D_\tau| = O\left(\frac{|B|^{1/2}|Q|^{1/2}}{|R|}\right) = O\left(\frac{|B|^2|Q|^2|R|^2}{|B|^{3/2}|Q|^{3/2}|R|^3}\right). \tag{19}$$

Note that $t \leq |R|$, $\tau \leq |B|$, and $\tau \leq |A| \leq |Q|$. Plugging these bounds into the denominator of (19) and taking $\beta$ to be sufficiently large implies (15).

Finally, assume that the bound of (18) is dominated by the term $|R||D_\tau|$. In this case, combining (16) and (18) gives $\tau|D_\tau| = O(t^{-1}|R||D_\tau|)$; that is, $\tau^2 t^2 = O(|R|^2)$. By also recalling the assumption $t^2\tau^2 > \beta|Q||R||B|$, we obtain $\beta|Q||R||B| = O(|R|^2)$, or $|Q||B| = O(|R|/\beta)$. Since we assume that $|B|$ is sufficiently large, this is a contradiction to $|Q| \geq |R|$. Thus, this case cannot occur. □

We will require the following result of Shkredov [7].

**Theorem 4.2.** *Let $A$ be a finite set of real numbers. Then*

$$|A+A| = \Omega\left(|A|^{79/37} c(A)^{-21/37}\right). \tag{20}$$

# 5 The improved sum-product bound

We are finally ready to prove Theorem 1.2. First, we note the *Katz-Koester inclusion* (see [2]). For any $\lambda \in A/A$ and $a = b \cdot c \in A_\lambda A_\lambda$, we have $b/\lambda \in A$. That is, $a = b \cdot c = \lambda c \cdot b/\lambda \in \lambda AA$, which in turn implies $a \in (AA) \cap (\lambda AA)$. Since this holds for every $a \in A_\lambda A_\lambda$, we have

$$A_\lambda A_\lambda \subset (AA) \cap (\lambda AA). \tag{21}$$

**Theorem 1.2.** *Consider a finite set $A \subset \mathbb{R}$. Then for any $c < 5/9813$ we have*

$$\max\{|A+A|, |AA|\} = \Omega^*(|A|^{4/3+c}).$$

*Proof.* We first claim that it suffices to prove the theorem for the case where $A$ consists of *positive* real numbers. Indeed, assume that the theorem applies to sets of positive real numbers and consider an arbitrary set $A$. We set $A' = A \setminus \{0\}$. If at least half of the elements of $A'$ are positive, let $A''$ be the set of positive elements of $A'$. Otherwise, let $A''$ be the set of absolute values of the negative elements of $A'$. Either way, $A''$ is a set of at least $\frac{|A|-1}{2}$ positive real numbers. We can then apply the theorem to $A''$, noting that $\max\{|A+A|, |AA|\} \geq \max\{|A''+A''|, |A''A''|\}$.

We now assume that $A$ is a set of positive real numbers. Let $r, r' \geq 1$ be real numbers such that $|A+A|^2|AA| \leq r|A|^4$ and $|AA|^3 \leq r'|A|^4$. We prove the theorem by showing that it is impossible for both $r$ and $r'$ to be asymptotically smaller than $|A|^{3c}$ and for $|A+A|$ to be asymptotically smaller than $|A|^{4/3+c}$.

We apply Lemma 3.2 with $A$ to obtain a real number $t$ and a set $\hat{A}'_t \subset \hat{A}_t$ with the properties that are stated in that lemma. Specifically, for every $\lambda \in \hat{A}'_t$, we have $|A_\lambda A_\lambda| = \Omega^*(t^2/r^{16})$. We denote this lower bound as $t' = \Omega^*(t^2/r^{16})$. For any $\lambda \in \hat{A}'_t$, the inclusion in (21) implies

$$|(AA) \cap (\lambda AA)| \geq |A_\lambda A_\lambda| \geq t'.$$

This in turn implies that $\lambda \in P_{t'}(AA, AA)$. Since this holds for every $\lambda \in \hat{A}'_t$, we get that $\hat{A}'_t \subset P_{t'}(AA, AA)$. Since $\hat{A}'_t \subset \hat{A}_t$, we also have

$$\sum_{a \in A} |A \cap (a\hat{A}'_t)| = \sum_{\lambda \in \hat{A}'_t} |A \cap (\lambda A)| \geq t|\hat{A}'_t|.$$

By the pigeonhole principle, there exists $a \in A$ such that $|A \cap (a\hat{A}'_t)| \geq t|\hat{A}'_t|/|A|$. We take an arbitrary $a$ that satisfies this, and set $A' = A \cap (a\hat{A}'_t)$. Since $\hat{A}'_t \subset P_{t'}(AA, AA)$, we get that $A' \subset P_{t'}(aAA, AA)$. Considering (14) with $Q = aAA$, $R = AA$ and $t'$, gives

$$c(A') \leq \frac{|AA|^4}{(t')^3} = O\left(\frac{r^{48}|AA|^4}{t^6}\right).$$

Next, we apply Theorem 4.2 to obtain

$$\begin{aligned}
|A + A| \geq |A' + A'| &= \Omega\left(|A'|^{79/37} c(A')^{-21/37}\right) \\
&= \Omega\left(\left(\frac{t|\hat{A}'_t|}{|A|}\right)^{79/37} \left(\frac{r^{48}|AA|^4}{t^6}\right)^{-21/37}\right) = \Omega\left(\frac{|\hat{A}'_t|^{79/37} t^{205/37}}{|A|^{79/37} r^{1008/37} |AA|^{84/37}}\right).
\end{aligned}$$

By Lemma 3.2 we have that $|\hat{A}'_t| = \Omega^*(E^\times(A)/t^2)$. This in turn implies

$$|A + A| = \Omega^*\left(\frac{(E^\times(A)/t^2)^{79/37} t^{205/37}}{|A|^{79/37} r^{1008/37} |AA|^{84/37}}\right) = \Omega^*\left(\frac{E^\times(A)^{79/37} t^{47/37}}{|A|^{79/37} r^{1008/37} |AA|^{84/37}}\right).$$

By Lemma 3.2 we have that $t > E^\times(A)/(2|A|^2)$, which gives

$$|A + A| = \Omega^*\left(\frac{E^\times(A)^{79/37} (E^\times(A)/|A|^2)^{47/37}}{|A|^{79/37} r^{1008/37} |AA|^{84/37}}\right) = \Omega^*\left(\frac{E^\times(A)^{126/37}}{|A|^{173/37} r^{1008/37} |AA|^{84/37}}\right).$$

Combining this with (2) yields

$$|A + A| = \Omega^*\left(\frac{(|A|^4/|AA|)^{126/37}}{|A|^{173/37} r^{1008/37} |AA|^{84/37}}\right) = \Omega^*\left(\frac{|A|^{331/37}}{r^{1008/37} |AA|^{210/37}}\right).$$

Since $|AA|^3 \leq r'|A|^4$, we conclude that

$$|A+A| = \Omega^*\left(\frac{|A|^{331/37}}{r^{1008/37}(r'|A|^4)^{70/37}}\right) = \Omega^*\left(\frac{|A|^{51/37}}{r^{1008/37}r'^{70/37}}\right).$$

If $\max\{r^{1/3}, r'^{1/3}\} = O^*(|A|^{5/3271})$ then the above gives $|A+A| = \Omega^*(|A|^{4/3+5/9813})$. Otherwise we have $\max\{r^{1/3}, r'^{1/3}\} = \Omega^*(|A|^{5/3271})$, which concludes the proof of the theorem. □

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